Topological Quantum Computation

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1 Introduction

Quantum information theory has significant practical implications for modern computing. Quantum algorithms leverage quantum parallelism and quantum interference to exhibit exponential speedup in probabilistic algorithms in comparison to their classical counterparts. Many fundamental algorithms have quantum realisations including: factorisation (exponential speedup), solving well-conditioned linear systems (exponential speedup), list search (polynomial speedup), and semi-definite programming (exponential speedup). The centrality of these algortihms to key challenges in modern computing (cyber security and deep learning to name a few) promote the study of physical models and realisations of quantum computing devices to research areas of significant interest and importance.

A particular challenge physical quantum computing devices are required to overcome is that of errors introduced by the physical realisation of quantum computation. Quantum mechanical systems are fragile and susceptible to errors through unwanted interactions with the environment. One approach to overcome the issue introduced by errors induced by undesired environmental interactions is called *quantum error correction*. The fundamental mathematical idea behind quantum error correction is to tensor our input qubits with ancillary qubits. One can then define a recovery operator which pushes errors into the ancillary qubits, and this error is then discarded by tracing out these qubits [KLM07]. An alternative approach is to define a model of computation which is robust to the small perturbations which may be introduce by environmental interactions. Topological quantum computation describes a quantum computing model for which the evolution of the quantum states are encoded by topological operations [FS18][LP17][LK06]. These topological operations are robust to local perturbations and thus if appropriately implemented would make the hardware robust to environmental interactions and less susceptible to errors. In this short essay I will briefly introduce the mathematical formulation of these topological quantum computations and give an example of a topological model which exhibits a two-qubit system.

2 Anyon Models

The physical objects of interest for topological quantum computation are called *anyons*. The state of an anyon is evolved by moving the point like particle along a loop. Within this model a non-trivial evolution is encoded by transporting the anyon along a non-contractible loop, and homotopic loops encode the same evolution. Anyons are quasiparticles which can be modelled to exist in two-dimensional spaces (such as graphene or lattices of cold atoms). The two spatial dimensions are key to non-trivial evolutions since in three spatial dimensions any loop of an anyon about other particles is null-homotopic, whilst in two dimensions the group of such loops is non-trivial. The groups describing the possible evolutions of particles in two dimensions are known as the braid groups. The elements of these groups can be represented diagrammatically as braids.

We make three assumptions for a physical system of anyons in order to formalise a mathematical framework to explore topological quantum computation:

1. Anyons are created or annihilated in pairs
2. Anyons can be fused to form other anyons
3. Anyons can be exchanged adiabatically

Thus in order to define an anyon model we are simply required to define our collection of anyon particles, the way in which these particles can fuse, how the states of these anyons evolve under adiabatic exchange (how the anyons evolve when traversing non-trivial loops around each other).

Let us give a set of labels to the distinct anyon particles in our model:

$$M = \{1, a, b, c, ...\}$$

The label 1 is the trivial label for no anyons present, and all other labels indicate distinct *topological charges.*

Definition 1. *(Fusion Rules) Let M be a collection of labels then a map $N : M \times M \to \mathbb{N}^M$ describes fusion rules for an anyon model, we require that for all $a \in M$ there is some $b \in M$ with $N^1_{ab} = 1$. We may write the fusion rule for a pair of anyons a, b as:*

$$a \times b = \sum_{c \in M} N^c_{ab} c$$

The fusion rules describe the possible particles which can be produced by fusing a pair of anyons. Non-zero coefficients N^c_{ab} indicate that it is possible for anyons a and b to fuse to form c. Note that we impose the condition that every particle has an anti-particle.

Definition 2. *(Fusion Space) Let M be a collection of labels with fusion rules N, the fusion space of a pair of anyons a, b is the vector space spanned by orthonormal vectors $\{|ab; c\rangle : N^c_{ab} \neq 0\}$*

The fusion space is the space of possible ways a pair of anyons may be fused. Since our fusion rules only dictate pairwise interaction we track a set of consistency relations which ensure associativity up to a change of basis for the fusion of multiple anyons. These relations are known as F-matrices. An F-matrix is a unitary matrix acting on a fusion space. Suppose states a, b, c can only fuse to state f. We can fuse a, b, c in two distinct ways. Either fusing a with b first to form d and then fusing d with c to form f, or alternatively fusing b with c first to form e and then fusing a with e to form f. These distinct choices of fusion space bases are related by the unitary matrix F^f_{abc} as follows:

$$|(ab)c; dc; f\rangle = \sum_e (F^f_{abc})_{de} \, |a(bc); ae; f\rangle$$

The F-matrices must satisy consistency relations known as the pentagon relation pictured in Figure 1, for which we represent the fusion order with a rooted tree, with leaves corresponding to

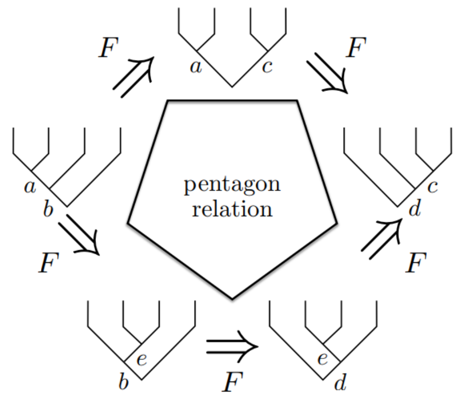


Figure 1: [FS18] The pentagon relation showing the consistency relations the F-matrices must satisfy.

the anyons we wish to fuse. We require that the two paths of associativity relations between fusing four anyons from left to right to fusing four anyons from right to left are equal.

To complete the description of an anyon model we need to describe the adiabatic exchanges between particles. We shall define exchange operators known as R-matrices. The unitary matrix R_{ab} describes the phase change induced by exchanging anyons a and b in a clockwise fashion. R_{ab} is diagonal acting on the fusion space of a, b with diagonal entries $R^c_{ab} = e^{i\theta^c_{ab}}$, a phase for each fusion channel. The effect of anti-clockwise exchange is given by the adjoint of this matrix.

Using the F-matrices and R-matrices we can exhibit all possible evolutions of exchanges. These exchanges must satisfy consistency criteria know as the hexagon relation, Figure 2. This relation is best visualised using braids. We require that exchanges and changes in fusion order which have the same effect on a braid are equal.

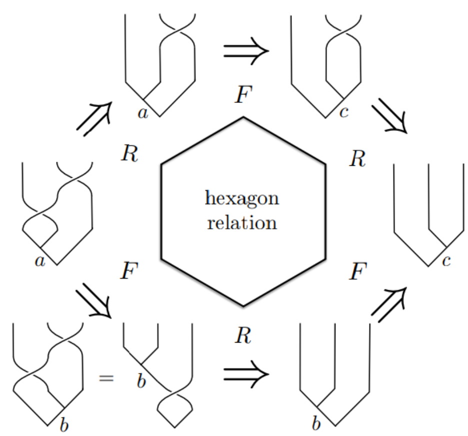


Figure 2: [FS18] The hexagon relation showing the consistency relations the R-matrices and F-matrices must satisfy.

3 Ising Anyon Model

In this section we shall introduce a simple anyon model which can be used to encode quantum computation with qubits. We shall see how to formulate a two qubit system and the corresponding braids for the X,Z and Hadamard gates as well as a controlled Z gate.

The Ising anyon model has the following set of particles and fusion rules:

$$M = \{1, \sigma, \psi\}$$

$$1 \times 1 = 1,\ 1 \times \psi = \psi,\ 1 \times \sigma = \sigma,$$
$$\psi \times \psi = 1,\ \sigma \times \sigma = 1 + \psi,\ \psi \times \sigma = \sigma$$

We shall refer to the particles σ as anyons and the particles ψ as fermions.

Examining the fusion rules for the Ising model we observe that the only non-deterministic fusion rule is given by fusing a pair of anyons. The fusion space of $2n$ anyons for a fixed channel can be deduced to be of dimension 2^{n-1}, since the first $n-1$ fusions are free to take either state and the final pair must fuse to achieve the correct parity of fermions to match the chosen channel.

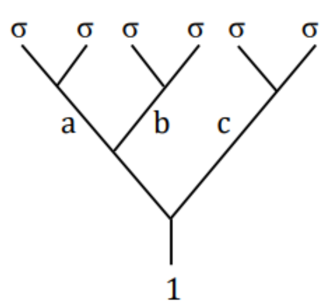
For a complete description of this model we need to give the F-matrices and R-matrices. For our purposes we only need give the F-matrix for fusing three anyons which is given by:

$$F = F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The R-matrix for the clockwise exchange of the two left most anyons in the two dimensional fusion space of three anyons is given by:

$$R = R_{\sigma\sigma} = \begin{pmatrix} R^{1}_{\sigma\sigma} & 0 \\ 0 & R^{\psi}_{\sigma\sigma} \end{pmatrix} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$$

Let us show how one can use a system of three pairs of anyons to encode a two qubit system. We shall use the vacuum channel of the fusion space to encode the four dimensional two qubit system.



	a	b	c
|00⟩	1	1	1
|10⟩	ψ	ψ	1
|01⟩	1	ψ	ψ
|11⟩	ψ	1	ψ

Figure 3: [LP17] Each basis element of the two qubit system is represented an element of the orthonormal basis of the fusion space for six anyons in the vacuum channel.

Let R_{ij} denote the clockwise exchange of anyons i and j. Consider the X gate acting on the first qubit, we claim that $X \otimes \mathrm{id} = (R_{23})^2$. Let us verify this identity computationally using the F-matrix and R-matrix described above.

The fusion diagram of Figure 3 specifies the following fusion order: $(((\sigma\sigma)(\sigma\sigma))\,\sigma\sigma)$. In order to exchange anyons the second and thirds anyons we change basis using $F \otimes \mathrm{id}$ to the fusion order: $((\sigma(\sigma(\sigma\sigma)))\,\sigma\sigma)$. The second and third anyons are now the leftmost two anyons of the first three anyons to be fused. Thus applying $R^2 \otimes \mathrm{id}$ exchanges the second and third anyons twice. Finally applying $F^{-1} \otimes \mathrm{id}$ restores the original fusion order. Hence we see that $(R_{23})^2 = F^{-1}R^2F \otimes \mathrm{id}$. A simple calculation yields:

$$F^{-1}R^2F = e^{-i\frac{\pi}{4}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Our calculation shows that the braid exchanging the second and third anyon twice is a topological operation which performs the X gate on the first qubit (up to constant phase factor).

Figure 4 below exhibits the braids for further operators:

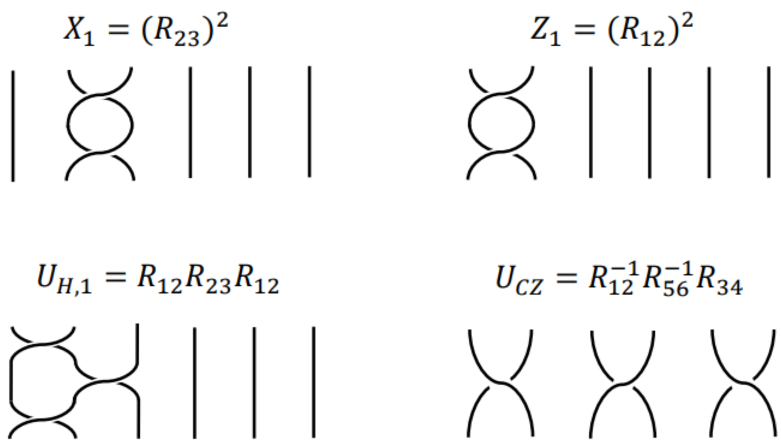


Figure 4: [LP17] The braids corresponding to the X, Z and Hadamard gate on the first gate, and the controlled Z gate.

We have seen how to perform the familiar X, Z, Hadamard gates and a two-qubit gate with topological operations. However this is not a universal gate set since we have not seen how to achieve a phase gate. This is a limitation of this particular model rather than a fundamental limitation of topological quantum computation.

References

[FS18] Bernard Field and Tapio Simula. Introduction to topological quantum computation with non-Abelian anyons. *Quantum Science and Technology*, 3:045004, Oct 2018.

[KLM07] Phillip Kaye, Raymond Laflamme, and Michele Mosca. *An Introduction to Quantum Computing*. Oxford University Press, Inc., New York, NY, USA, 2007.

[LK06] Jr. Lomonaco, Samuel J. and Louis H. Kauffman. Topological quantum computing and the Jones polynomial. In *Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series*, volume 6244 of *Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series*, page 62440Z, May 2006.

[LP17] Ville Lahtinen and Jiannis Pachos. A Short Introduction to Topological Quantum Computation. *SciPost Physics*, 3:021, Sep 2017.