Problem Sheet 1

Morse Theory
TT 2019

Oliver Vipond

1 Manifolds

Problem 1.1. (a) *Let M be a compact manifold with finite atlas $\{(U_i, \varphi_i)\}_{i=1}^N$. Show that there is an open cover of M by open sets $\{V_i\}_{i=1}^N$ such that $\bar{V}_i \subset U_i$*

(b) *Suppose there exist functions $f_i : M \to \mathbb{R}_{\geq 0}$ with support contained in U_i and such that $f_i(V_i) = 1$. Deduce there exists functions $p_i : M \to \mathbb{R}_{\geq 0}$ with support contained in U_i and such that $\sum_i p_i = 1$*

Problem 1.2. (a) *Suppose $f : M \to N$ is an immersion at $p \in M$ so that $d_p f$ in injective. Show that there are local coordinates near p and $f(p)$ such that*

$$f(x_1, ..., x_n) = (x_1, ..., x_n, 0, ..., 0)$$

(b) *Let $f_1, ..., f_n : M \to \mathbb{R}$ be such that the set $\{d_p f_i\}$ is linearly independent. Show that $V := \{f_1 = ... = f_n = 0\} \subset M$ is a submanifold of codimension n locally about p. What is the tangent space of this submanifold at p?*

(c) *Suppose $\iota : N \to M$ is an embedded submanifold. Show that N is locally a complete intersection, that is to say at each point $p \in N$ the set of derivatives $\{d_p \iota_j\}$ is linearly independent.*

(d) *Show that any compact submanifold $N \subset M$ can be realised as the zero level set of some $f : M \to \mathbb{R}^k$*

Problem 1.3. *State Sard's Theorem and give a sketch proof of this result*

2 Algebraic Topology

Problem 2.1. (a) *Let C_* be a chain complex and H_* the homology of this complex. Is C_* always chain homotopy equivalent to H_*?*

(b) *Give an example of a chain map $f_* : C_* \to D_*$ whish is a quasi-isomorphism (induces isomorphism on homology groups), but is not a chain homotopy equivalence*

Problem 2.2. *(a) Let the Klein bottle K be given the CW complex structure of a square $[0,1]^2$ with side identifications $\{(x,0)=(x,1)\},\{(0,y)=(1,1-y)\}$. Let $f : K \to K$ be the map induced by a π rotation of K about the midpoint. Determine the induced maps in first homology with both integral and $\mathbb{Z}_2$ coefficients.*

(b) Consider the mapping torus T_f and compute the homology of T_f with integral coefficients.

(c) Deduce whether T_f is an orientable manifold.

3 Functional Analysis

Problem 3.1. *Let $L : E \to F$ be a linear operator between Banach spaces. We say L is Fredholm if the kernel and cokernel are finite dimensional. The index of such an operator is taken to be $Ind(L) = \dim kerL - \dim cokerL$. Let L, L_0, L_1 be Fredholm operators, show that the following are Fredholm and compute their indices:*

(a) $L_0 \oplus L_1$

(b) $L \otimes id_V$ *for a finite dimensional vector space* V

(c) $L_1 \circ L_0$

Problem Sheet 2

Morse Theory
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1 Morse Functions

Problem 1.1. *Let $f : M \to \mathbb{R}$ have a critical point at $p \in M$ i.e $d_p f = 0$. Define the Hessian of f at p to be the bilinear form $H(f) : T_pM \times T_pM \to \mathbb{R}$ given by the formula $H_p(f)(X,Y) = X(\tilde{Y}(f))$ where $\tilde{Y}$ is an arbitary extension of Y to a neighbourhood of p.*

(a) Show that $H_p(f)$ is well-defined and symmetric

(b) What is the matrix representation of the Hessian in a local chart?

Problem 1.2. *Consider a compact manifold $M \subset \mathbb{R}^n$. Let $dist_p(x) = \|x - p\|^2$ denote the distance function to the point p.*

(a) Show that for almost all $p \in \mathbb{R}^n$ the function $dist_p(x)$ is Morse

*(b) Consider the torus $\{(x_1, x_2, x_3) : \left(R - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 = r^2\} \subset \mathbb{R}^3$. For which $p \in \mathbb{R}^3$ is $dist_p(x)$ **not** a Morse function?*

Problem 1.3. *Consider a manifold $M \subset \mathbb{R}^n$ equipped with a function $f \in \mathcal{C}^\infty(M, \mathbb{R})$ not necessarily Morse.*

(a) Show that f and its first k derivatives can be uniformly approximated on any compact subset of M by a Morse function.

2 Critical Points

Problem 2.1. *Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For which $n \in \mathbb{N}$ can you produce a Morse function on $\mathbb{T}^2$ with n critical points?*

Problem 2.2. *Let $f : M \to \mathbb{R}$ be a Morse function with finitely many critical points. Let C_* be a chain complex of vector spaces over $\mathbb{F}$.*

Define the Morse polynomial of f to be:

$$P_f(t) := \sum_{p \in Crit(f)} t^{\lambda(p)}$$

Define the Poincaré series of C_* to be:

$$P_{C_*}(t) := \sum_k \dim_{\mathbb{F}}(C_k) t^k$$

Define the Poincaré polynomial of M to be:

$$P_M(t) := P_{H_*(M,\mathbb{F})}(t) = \sum_k \beta_k(M) t^k$$

(Where $\beta_k(M) := \dim_{\mathbb{F}} H_k(M, \mathbb{F})$ denotes the k^{th} Betti number of M.)
Define an order relation on the ring $\mathbb{Z}[t, t^{-1}]$ given by:

$P(t) \succ R(t) \iff$ *there exists $Q(t) \in \mathbb{Z}_{\geq 0}[t, t^{-1}]$ such that $P(t) = R(t) + (1 + t)Q(t)$*

(a) *Given a long exact sequence $... \to A_k \to B_k \to C_k \to A_{k-1} \to ...$ show that*

$$P_{A_*} + P_{C_*} \succ P_{B_*}$$

(b) *Show that $P_f \succ P_M$*

(c) *If $P_f = P_M$ what does this mean in terms of the Morse inequalities?*

Problem 2.3. *Let M be a compact, orientable 3-manifold equipped with Morse function f. Suppose M has integral homology isomorphic to S^3.*

(a) *Show that f must admit an even number of critical points.*

(b) *Construct a Morse function on $S^1 \times S^2$ with precisely 4 critical points.*

(c) *Show that if $\pi_1(M) \neq 1$ then f must have at least 6 critical points.*

Problem Sheet 3

Morse Theory
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1 Transversality

Problem 1.1. *For a smooth map $f : M \to N$, we say that f is transverse to a submanifold $Q \subset N$ if for all $q \in Q$ and for all $p \in f^{-1}(q)$ we have $d_p f(T_p M) + T_q Q = T_q N$, and we write $f \pitchfork Q$.*

We say that a property $\mathcal{P}$ of maps $f : M \to N$ is stable if given any homotopy $f_t : M \times I \to N$ such that f_0 has property $\mathcal{P}$, then there exists an $\varepsilon > 0$ such that f_t satisfies property $\mathcal{P}$ for all $t < \varepsilon$.

(a) *Show that $f \pitchfork Q \implies f^{-1}(Q) \subset M$ is a submanifold. What is the codimension of $f^{-1}(Q) \subset M$?*

(b) *Let M be compact. Show that the following properties of maps $f : M \to N$ are stable:*

- *f a local diffeomorphism*
- *$f \pitchfork Q$ for fixed $Q \subset N$*

Problem 1.2. *Study the start of Chapter 3 of Morse Theory and Floer Homology on the Morse Complex.*

(a) *Sketch the genus-2 surface equipped with a height function.*

(b) *Define the stable and unstable manifolds associated to a critical point. Sketch these submanifolds for a selection of critical points on the genus-2 surface.*

(c) *Compute the Morse complex (both chain groups and differentials)*

2 Poincaré Conjecture

Problem 2.1. *We will show any simply connected, closed manifold M with dimension $m \geq 6$ which has the same homology as a sphere is homeomorphic to the sphere.*

(a) *Let $X = M \setminus D_\epsilon$ be the manifold with a small open m-disc removed. Show that $H_0(X) \cong H^m(M, D_\epsilon) \cong \mathbb{Z}$ and the (co)homology groups are zero in all other degrees*

(b) *Show X is simply connected, compact, smooth manifold with simply connected boundary ∂X*

[Hint: Appeal to the fact that transversality is an open condition]

(c) Show the above conditions hold if we remove another small open m-disc from the interior of X.

(d) Look at the h-cobordism theorem. What does this tell us about X?

(e) Deduce that M can be decomposed into two nice spaces glued by a nice morphism. Use this to deduce a homeomorphism to S^m.

3 Connections

Problem 3.1. *Let $E \to M$ be a vector bundle over M with connection ∇ and $\gamma : [0,1] \to M$ a smooth path. Let γ^*E denote the pull back bundle and $\gamma^*\nabla$ the pull back of the connection: $\gamma^*\nabla : \mathcal{C}^\infty(T[0,1]) \otimes \mathcal{C}^\infty(\gamma^*E) \to \mathcal{C}^\infty(\gamma^*E)$. We will abbreviate $(\gamma^*\nabla)(\partial_t \otimes v)$ by $\nabla_t v$.*

(a) The parallel transport along γ is a linear map $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$, mapping $v_0 \in E_{\gamma(0)}$ to $P_\gamma(v_0) = v(1)$ where v is the unique solution to the linear ODE:

$$\nabla_t v = 0, v(0) = v_0$$

Show that P_γ is an isometry when ∇ is the Levi-Civita connection.

(b) Deduce that geodesics have constant speed

Problem Sheet 4

Morse Theory
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1 Properites of Morse Homology

Problem 1.1. *We have seen in Chapter 3 of Morse Theory and Floer Homology that the Morse homology is independent of our choice of Morse-Smale function. For each of the following familiar properties of cellular homology show that Morse homology also enjoys this property. In each case provide a Morse-Smale function which exhibits the appropriate correspondence between critical points and differentials. [Attempt this question before reading Chapter 4!]*

(a) Poincaré Duality

(b) Künneth Formula

2 Stable Manifold Theorem

Problem 2.1. *Let M be a manifold with Morse function f. Recall that a metric on M is a symmetric, positive definite, bilinear form.*

(a) Show that the collection of metrics forms a convex set

(b) Using the Morse Lemma deduce that there exists a metric g on M such that for each critical point p there is a neighbourhood of p with local coordinates $(x_1, ..., x_n)$ such that the gradient flow equation $(\frac{d}{dt}\gamma(t) = -grad_{\gamma(t)} f)$ takes the form $\dot{x}_i = a_i x_i$ for real constants $a_i \in \mathbb{R}$. We say such a metric is 'nice'.

(c) Suppose M is equipped with a nice metric. Show that the stable manifold of a critical point of index λ us diffeomorphic to an open disc of dimension $n - \lambda$

3 Lefschetz-Hyperplane Theorem

Read and digest Section 7 of Milnor on *The Lefschetz Theorem on Hyperplane Sections*. We shall prove interesting implications for hypersurfaces in $\mathbb{CP}^3$.

Problem 3.1. *(a) Let $\{m_i\}_{i=0}^N$ be the collection of monomials of degree d in variables $x_0, ..., x_n$. Compute N.*

(b) *The Veronese embedding of degree d,* $V_d : \mathbb{CP}^n \to \mathbb{CP}^N$*, is given by the map:*

$$V_d([x_0 : ... : x_n]) = [m_0 : ... : m_N]$$

The image $V_d(\mathbb{CP}^n)$ *is known as the Veronese variety. Show that a hypersurface of* $\mathbb{CP}^n$ *may may be realised as a hyperplane section of the Veronese variety.*

(c) *Use the Lefschetz Hyperplane Theorem to deduce that a hypersurface of* $\mathbb{CP}^3$ *is simply connected.*

(d) *Show that if a 4-manifold is simply connected then it possesses no torsion in homology.*

(e) *For a smooth hypersurface of degree d,* $X_d \subset \mathbb{CP}^{n+1}$ *the Euler characteristic of* X_d *is given by:*

$$\chi(X_d) = n + 2 + \frac{1}{d}[(1-d)^{n+2} - 1]$$

Show that the quartic surface $X_4 \subset \mathbb{CP}^3$ *has* $H_2(X_4) \cong \mathbb{Z}^{22}$

Problem Sheet 5

Morse Theory
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1 The Energy Functional and Jacobi Fields

Problem 1.1. *Let $\overline{\alpha} : (-\varepsilon, \varepsilon) \to \Omega M$ denote a variation of $\omega \in \Omega M$ in direction $W \in T_\omega \Omega M$. Recall the length function and energy function:*

$$L(\omega) = \int_0^1 \|\dot{\omega}(t)\| dt, \quad E(\omega) = \int_0^1 \|\dot{\omega}(t)\|^2 dt$$

(a) *Compute the first variation of the length function and show:*

$$\left.\frac{dL(\overline{\alpha}(u))}{du}\right|_{u=0} = -\int_0^1 \langle W_t, \tilde{A}_t \rangle dt - \sum_{i=0}^{k} \langle W_{t_i}, \tilde{V}_{t_i^+} - \tilde{V}_{t_i^-} \rangle$$

Where $\tilde{V}_t = \frac{\dot{\omega}(t)}{\|\dot{\omega}(t)\|}$ is the normalised velocity and $\tilde{A}_t = \nabla_{\dot{\omega}} \tilde{V}_t$ the acceleration of the normalised velocity.

(b) *By considering the first variation of the energy functional, show that if a piecewise smooth path is critical for the energy functional then the path must be $\mathcal{C}^1$ and hence a geodesic.*

Let $R : \bigotimes_{i=1}^3 \mathcal{C}^\infty(TM) \to \mathcal{C}^\infty(TM)$ be the trilinear assignment:

$$R(X,Y)Z = -\nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) + \nabla_{[X,Y]} Z$$

A vector field X along a geodesic γ is a Jacobi field if:

$$\nabla_{\dot{\gamma}}^2 X + R(\dot{\gamma}, X)\dot{\gamma} = 0$$

(c) *Which of the following three vector fields are Jacobi fields along γ:*
$X_0 = \dot{\gamma}$, $X_1 = t\dot{\gamma}$, $X_2 = t^2\dot{\gamma}$

(d) *Show that for a Jacobi field X, $\langle X, \dot{\gamma} \rangle$ is a linear function along γ*

2 Heegard Decomposition of a 3 Manifold

Problem 2.1. *We will show that a compact connected 3-manifold M can be decomposed into a pair of handlebodies (where a handlebody is a 3-ball with 1-handles attached).*

(a) Let f be a self-indexing Morse-Smale function on M. By considering an appropriate sublevel set of f, show that M has a submanifold which is a handlebody. Be sure to show that the sublevel set is connected.

(b) Deduce that a corresponding superlevel set of f is also a handlebody, and thus M admits a decomposition into a pair of handlebodies.

3 Freudenthal Suspension Theorem

Review Section 15-17 of Milnor, in particular the computation of the homotopy type of ΩS^n. Let Σ denote the reduced suspension functor, Ω the loop functor, and $[X, Y]$ homotopy classes of maps from X to Y.

Problem 3.1. *(a) Describe a bijection $[\Sigma X, Y] \cong [X, \Omega Y]$ and briefly explain why it is well-defined.*

(b) Using the Morse Theory computation of the homotopy type of ΩS^n, deduce that:

$$\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1}) \text{ for } k+2 \leq n$$

Problem Sheet 6

Morse Theory
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1 Discrete Morse Theory

Problem 1.1. *Let K be a simplicial complex and $\sigma^{(p)}$ denote a simplex of dimension p. Suppose $f : K \to \mathbb{R}$ has the property that for all $\sigma^{(p)} \in K$:*

$$|\{\tau^{(p-1)} \in K | \tau^{(p-1)} < \sigma^{(p)}, f(\tau^{(p-1)}) \geq f(\sigma^{(p)})\}| \leq 1$$

$$|\{\tau^{(p+1)} \in K | \tau^{(p+1)} > \sigma^{(p)}, f(\tau^{(p+1)}) \leq f(\sigma^{(p)})\}| \leq 1$$

(a) Show that for each simplex at least one of the above inequalities is strict.

(b) Deduce that such a function induces a gradient vector field and show that the induced vector field is acyclic.

Problem 1.2. *Choose a triangulation of the 2-torus and find a perfect discrete Morse function on your chosen triangulation. Sketch the gradient vector field induced by your function.*

Problem 1.3. *We say a topological space is* ***simply contractible*** *if it is homeomorpic to a simplicial complex which admits a finite sequence of elementary collapses to a single vertex. Show that the Dunce hat is contractible, but not simply contractible.*

Problem 1.4. *We say a discrete Morse function is a* ***collapsing scheme*** *if it has precisely one critical point.*

(a) Let $f : K \to \mathbb{R}$ be a collapsing scheme on K. Show that the critical simplex is a vertex, and f attains its minimum on this simplex.

(b) Suppose that K has an odd number of simplices. Show that K admits a collapsing scheme if and only if K is simply contractible

Problem 1.5. *Let $\Delta_n^{(k)}$ denote the k-skeleton of the n-simplex.*

(a) Show that $\Delta_3^{(k)}$ is homotopic to a wedge of k-spheres for $k = 0, 1, 2, 3$

(b) Use discrete Morse theory to deduce the homotopy type of $\Delta_n^{(k)}$ for all n, k

Problem Sheet 7

Morse Theory
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1 Symplectic Geometry

Problem 1.1. *Consider the symplectic manifold* $(\mathbb{R}^{2n+2}, \omega = \sum dp_i \wedge dq_i)$ *and consider the function* $f : \mathbb{R}^{2n+2} \to \mathbb{R}$ *given by* $f(p_1, ..., p_{n+1}, q_1, ..., q_{n+1}) = \sum p_i^2 + q_i^2$

(a) *Compute the associated Hamiltonian vector field* X_f *and Hamilton equations*

(b) *Give an explicit description of the associated family of diffeomorphisms*

Problem 1.2. *Let* $(p_1, ..., p_n)$ *be local coordinates on some* $U \subset M$*. Let the cotangent bundle have respective local coordinates* $(p_1, ..., p_n, q_1, ..., q_n)$*. Let* α *be the 1-form on* T^*M *given pointwise by:*

$$\alpha_{(p,q)} = (d\pi_{(p,q)})^* q$$

Show that on T^*U*,* $\alpha = \sum q_i dp_i$

Problem 1.3. *Recall the Hamiltonian system equation:* $\dot{x}(t) = X_{H_t}(x(t))$*. Let* $\mathcal{L}M$ *denote the free loop space of a symplectic manifold* (M, ω)*. Recall that an element of the tangent space at an element of the loop space* x*, is a vector field* ξ *along the loop* x*.*

(a) *Show that if* H_t *is time independent then* $H = H_t$ *is constant on a trajectory satisfying the Hamiltonian system equation*

(b) *Let* H_t *be a 1-periodic Hamiltonian* $(H_t = H_{t+1})$*. Consider the 1-form* $\Psi_H : T\mathcal{L}M \to \mathbb{R}$*:*

$$\Psi_H(x; \xi) = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t)), \xi(t))dt$$

Prove that this 1-form is closed.

(c) *Let* $\bar{x}$ *denote an extension of* $x(t)$ *to the 2-disc* D *(*x *contractible). Assume that the formula below gives a well defined function* $a_H : \mathcal{L}M \to \mathbb{R}/\mathbb{Z}$*:*

$$a_H(x) = -\int_D \bar{x}^*\omega - \int_0^1 H_t(x(t))$$

Show that the differential of a_H *is* Ψ_H